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Uniqueness Solution of the Finite Elements Scheme for Symmetric Hyperbolic Systems with Variable Coefficients

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ABSTRACT

The present work is devoted to the proof of uniqueness of the solution of the finite elements scheme in the case of variable coefficients. Finite elements method is applied for the numerical solution of the mixed problem for symmetric hyperbolic systems with variable coefficients. Moreover, dissipative boundary conditions and its stability are proved. Finally, numerical example is provided for the two dimensional mixed problem in simply connected region on the regular lattice. Coding is done by DELPHI7.

1. The Statement of the Problem ([2],[3])

There are many results [(D.Aloev et al., 2014),(Falk and Richter, 1999),(Du et al., 1988),(Hughes and Tezduyar, 1984),(Segerlind, 1976),(Adjerid and Baccouch, 2010) and (Codina, 2008)] that are devoted to the study of finite element method (FEM) for a mixed problem for symmetric hyperbolic systems with the help of finite differences method. Various explicit and implicit finite difference schemes (FDS) are obtained. The stability of the obtained schemes is also investigated. Error estimates of the approximate solutions of the mixed problems are given. Particularly, in the preceding work of the authors [1], the stability of the implicit difference scheme obtained by the finite elements method for two dimensional t - hyperbolic system with constant coefficient were proved. A detailed review of [(Falk and Richter, 1999),(Du et al., 1988),(Hughes and Tezduyar, 1984),(Segerlind, 1976),(Adjerid and Baccouch, 2010) and (Codina, 2008)] is given in [(D.Aloev et al., 2014)].

We consider the following t - hyperbolic system of the form

$$A(t,x,y)\frac{\partial u}{\partial t}+B(t,x,y)\frac{\partial u}{\partial x}+C(t,x,y)\frac{\partial u}{\partial y}+D(t,x,y)u=F(t,x,y), \quad (1)$$

in the domain

$$G=\Big\{(t,x,y): t\in(0,T), \quad (x,y)\in\Omega\Big\},$$

where

$$\Omega=\{(x,y): \quad 0<x<l_x, \quad |y|<\infty\}$$

with the boundary conditions:

For $x=0, \quad 0<t\leq T, \; |y|<\infty:$

$$u^I(t,0,y)=S(x,y)u^{II}(t,0,y)+g_1(x,y), \quad (2)$$

for $x=l_x, \quad 0<t\leq T, \; |y|<\infty:$

$$u^{II}(t,l_x,y)=R(x,y)u^I(t,l_x,y)+g_2(x,y), \quad (3)$$

for $|y|\to\infty$ and $0<t\leq T, \; 0\leq x\leq l_x:$

$$\|u\|\to 0, \quad \|u\| = \sqrt{\sum_{k=1}^{N}u_k^2} \quad (4)$$

and with the initial data

$$u(0,x,y)=u_0(x,y), \;\; (x,y)\in\Omega. \quad (5)$$

Here $\{t, x, y\}$ are independent variables and

$$B(t, x, y) = (b_{ks}(t, x, y)), \quad C(t, x, y) = (c_{ks}(t, x, y)),$$

are symmetric matrices of $N \times N$ dimension, the elements of these matrices are given functions, and A is the positive defined matrix; $D(t, x, y) = (d_{ks}(t, x, y))$ is the square matrix of $N \times N$ dimension and R, S are rectangle matrices, in which the number of rows are respectively equals to the number of the positive and negative eigenvalues of the matrix B.

In equations (1)-(5), the function

$$u(t, x, y) = (u_1, u_2, \cdots, u_N)^T$$

is unknown vector function and

$$F(t, x, y) = (f_1, f_2, \cdots, f_N)^T$$

is the given function.

Approximation of the domain G and the description of the finite elements methods is given in [(D.Aloev et al., 2014)]. High accuracy finite element methods are given in [(Hughes and Tezduyar, 1984),(Segerlind, 1976),(Adjerid and Baccouch, 2010) and (Codina, 2008)] for different type of hyperbolic equations.

The present work is devoted to the proof of uniqueness of the solution of the finite elements scheme in the case of variable coefficients. Finite elements method is applied for the numerical solution of the mixed problem for symmetric hyperbolic systems with variable coefficients. Moreover, dissipative boundary conditions and its stability are proved. Finally, numerical example is provided for the two dimensional mixed problem in simply connected region on the regular lattice. Coding is done by DELPHI7.

The paper is arranged as follows: In Section 2, construction of the difference scheme for symmetric t-hyperbolic system is described. Section 3 discusses the uniqueness of the solution of the finite elements scheme. Finally, numerical example is provided in Section 4 to show the accuracy and efficiency of the method. Section 5 concludes the main ideas of the the approximate method.

2. Construction of the Difference Scheme for Symmetric t-Hyperbolic System

We divide the domain Ω to the finite elements which have not generic interior points. The finite element is denoted by K. Then it is clear that

$$\Omega = \bigcup_{K \subset \Omega} K.$$

Next, the interval $[0, T]$ is divided into N_t parts and the partitioning points are

$$t_k = \tau \cdot n, (n = 0, ..., N_t), \tau = \frac{T}{N_t}.$$

In each time layer t_n the approximate solution $u_h\left(t_n, x, y\right)$ of the mixed problem (1)-(5) is searched in the form

$$u_h^n = u_h\left(t_n, x, y\right) = \sum_{(x_i; y_j) \in \Omega} u_{ij}^n Q_{ij}(x, y),$$

where $Q_{ij}(x, y)$ is the base function [5], its values at the nodes $(x_i, y_j) \in \Omega$ are equal to one, and in other points are equal to zero. Let

$$u_{ij}^n = u(x_i, y_j, t_n) = \begin{pmatrix} u_{1ij}(t_n) \\ u_{2ij}(t_n) \\ \vdots \\ u_{Nij}(t_n) \end{pmatrix} = \begin{pmatrix} u_{1ij}^n \\ u_{2ij}^n \\ \vdots \\ u_{Nij}^n \end{pmatrix}$$

On the element K with the nodes M_{ij} we approximate equation (1) by the implicit difference scheme:

$$\begin{aligned} &\left(A(t_{n+1}, x, y)\frac{u_h^{n+1} - u_h^n}{\tau}, Q_{ij}\right)_K + \left(B(t_{n+1}, x, y)\frac{\partial u_h^{n+1}}{\partial x}, Q_{ij}\right)_K \\ &+ \left(C(t_{n+1}, x, y)\frac{\partial u_h^{n+1}}{\partial y}, Q_{ij}\right)_K + \left(D(t_{n+1}, x, y)u_h^{n+1}, Q_{ij}\right)_K \\ &= \left(F(t_{n+1}, x, y), Q_{ij}\right)_K, \quad (x_i, y_j) \in \Omega, \end{aligned} \tag{6}$$

where $(u, v)_K = \iint\limits_K u(x, y) \cdot v(x, y) dK$.

Let us rewrite the difference scheme (6) for $(x_i, y_j) \in \Omega$ in the following form

$$\left(A(t_{n+1},x,y)u_h^{n+1}, Q_{ij}\right)_K + \tau\left(B(t_{n+1},x,y)\frac{\partial u_h^{n+1}}{\partial x}, Q_{ij}\right)_K + \tau\left(C(t_{n+1},x,y)\frac{\partial u_h^{n+1}}{\partial y}, Q_{ij}\right)_K + \tau\left(D(t_{n+1},x,y)u_h^{n+1}, Q_{ij}\right)_K = \tau\left(F(t_{n+1},x,y), Q_{ij}\right)_K + \left(A(t_{n+1},x,y)u_h^{n}, Q_{ij}\right)_K. \quad (7)$$

3. Uniqueness of the Solution of the Finite Elements Scheme

For simplicity we suppose that A is unit matrix. Assume that the mixed problem (1)-(5) has a unique solution and the boundary conditions satisfy the following conditions [4]:

$$\int\limits_{\Gamma(\Omega)} Su\cdot u ds \geq 0 \;\forall t\in[0,T]. \quad (8)$$

$$D + D^* - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y} \geq 0, \quad (9)$$

where $S = n_x B + n_y C$ and $n = (n_x, n_y)$ is the unit external normal to the Ω, and $\Gamma(\Omega)$ is the boundary of the domain Ω.
Consider the following differential-difference system:

$$Lu \equiv \tau B(t,x,y)\frac{\partial u}{\partial x}(t,x,y) + \tau C(t,x,y)\frac{\partial u}{\partial y}(t,x,y) + (I + \tau\cdot D(t,x,y))u(t,x,y) = u(t-\tau,x,y) + \tau\cdot F(t,x,y). \quad (10)$$

Here I is the unit matrix, t is the time divisibly by τ.

We consider the bilinear form

$$a(u,v) \equiv (Lu,v)_K, \quad (11)$$

where

$$(u,v)_K = \int\limits_K u\cdot v dK. \quad (12)$$

Let $P_m(K)$ be the set of polynomials of order $\leq m$ defined on K, the coefficients do not depend on t. Then for each element of K the following equality holds

$$a(u_h, v_h) = (u_h(t-\tau, x) + \tau \cdot F(t, x), v_h)_K, \tag{13}$$

where $\forall v_h \in P_m(K)$.

If the solution is known on the layer $t-\tau$ and $v_h \in P_m(K)$ is the base function, then from (13) and definition of the solution on the layer t we get the system of algebraic equations (7). Let us introduce the following operator:

$$\begin{aligned} L^* v \equiv v &- \tau B(t,x,y)\frac{\partial v}{\partial x}(t,x,y) \\ &- \tau C(t,x,y)\frac{\partial v}{\partial y}(t,x,y) + \tau (D^*(t,x,y) \\ &- \frac{\partial B}{\partial x}(t,x,y) - \frac{\partial C}{\partial y}(t,x,y))v(t,x,y). \end{aligned} \tag{14}$$

Lemma 1. *The following equality holds*

$$a(u,v) = (u, L^* v)_K + \tau \int\limits_{\Gamma(K)} Su \cdot v \tag{15}$$

Using integration by parts we obtain

$$\begin{aligned} a(u,v) &\equiv (Lu, v)_K \\ &= \left(\tau B \frac{\partial u}{\partial x} + \tau C \frac{\partial u}{\partial y} + (I + \tau \cdot D)u, v\right)_K \\ &= \tau \left(\int\limits_{\Gamma(K)} Su \cdot v \right) + ((I + \tau \cdot D^*)v, u)_K \\ &\quad - \left(B\frac{\partial v}{\partial x} + C\frac{\partial v}{\partial y}, u\right)_K - \left(\left(\frac{\partial B}{\partial x} + \frac{\partial C}{\partial y}\right) v, u\right)_K \\ &= \left(v - \tau B\frac{\partial v}{\partial x} - \tau C \frac{\partial v}{\partial y} + \tau\left(D^* - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}\right) v, u\right)_K + \tau \int\limits_{\Gamma(K)} Su \cdot v. \end{aligned}$$

Lemma is proved.

Lemma 2. *The following inequality is true*

$$a(u,u) \geq (u,u)_K + \frac{\tau}{2} \int\limits_{\Gamma(K)} Su \cdot u. \tag{16}$$

Doing some transformations we get the following chain of inequalities:

$$\begin{aligned}(u, L^*u)_K &= \left(u - \tau\left(B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial x}\right) + \tau\left(D^* - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}\right)u, u\right)_K \\ &= -\left(u + \tau\left(B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial y}\right) + \tau Du, u\right)_K \\ &\quad + \left(2u + \tau Du + \tau(D^* - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y})u, u\right)_K \\ &= -a(u,u) + \left(\left(2I + \tau\left(D + D^* - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}\right)\right)u, u\right)_K \end{aligned} \tag{17}$$

Taking into account (9), using equalities (15) and (17) we get the following inequality:

$$\begin{aligned} a(u,u) &= \frac{1}{2}\left((2I + \tau(D + D^* - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}))u, u\right)_K \\ &\quad + \frac{\tau}{2}\int\limits_{\Gamma(K)} Su\cdot u \geq (u,u)_K + \frac{\tau}{2}\int\limits_{\Gamma(K)} Su\cdot u. \end{aligned} \tag{18}$$

Theorem 1. *If the solution of the finite elements scheme of type $u_h \in P_n(K)$ exists and is convergent, then it is uniquely defined on K and satisfy the following inequality:*

$$\|u_h(t,x,y)\|_\Omega^2 \leq e^T \|u_h(0,x,y)\|_\Omega^2 + (T+1)(e^T - 1)F, \tag{19}$$

where $\|u\|_\Omega = \sqrt{\int\limits_\Omega u\cdot u}$, $\quad F = \max\limits_{t\in[0,T]} \|F_h(t,x,y)\|_\Omega^2$.

Assume that the values of the convergent solution $u_h^{k-1} \in P_m(K)$ on the layer $t_{k-1} = \tau(k-1)$ are defined and the base function $v_h \in P_m(K)$ is chosen, then for definition of the values of the convergent solution $u_h^k \in P_m(K)$ on the layer $t_k = \tau k$ from equality (11) we get the following system of linear algebraic equations :

$$\begin{aligned} &\tau\left(B(t_k,x,y)\frac{\partial u_h}{\partial x}(t_k,x,y) + C(t_k,x,y)\frac{\partial u_h}{\partial y}(t_k,x,y)\right)_K \\ &\quad + \left((I + \tau\cdot D(t_k,x,y))\,u_h(t_k,x,y), v_h\right)_K \\ &\quad = (u_h(t_{k-1},x,y) + \tau\cdot F_h(t_k,x,y), v_h)_K. \end{aligned} \tag{20}$$

We rewrite the system (20) in the matrix form:

$$A_h u_h = b_h \tag{21}$$

If we prove that the homogenous system of algebraic equations $A_h u_h = 0$ has only trivial solution $u_h \equiv 0$, then the uniqueness of the solution of the nonhomogeneous system (21) holds. This problem is equivalent to the proof of the equality $u_h = 0$ on K, i.e.

$$\begin{aligned}&\tau\left(B(t_k,x,y)\frac{\partial u_h}{\partial x}(t_k,x,y)+C(t_k,x,y)\frac{\partial u_h}{\partial y}(t_k x,y)\right)_K\\&+\Big((I+\tau\cdot D(t_k,x,y))u_h(t_k,x,y),v_h\Big)_K\\&=0,\quad \forall v_h\in P_n(K).\end{aligned}\tag{22}$$

We can take $v_h = u_h$, then taking into account the fact that the left part of equality (20) is $a(u_h, v_h)$. From Lemma 3.2 we will get the following inequality:

$$(u_h,u_h)_K+\frac{\tau}{2}\int\limits_{\Gamma(K)-\Gamma^*(K)} S_h u_h\cdot u_h ds+\frac{\tau}{2}\int\limits_{\Gamma^*(K)} S_h u_h\cdot u_h ds\le 0,\tag{23}$$

where $\Gamma^*(K) = \Gamma(K) \cap \Gamma(\Omega)$.

Because of arbitrariness of the element $K \in \Omega_h$ from (23) we obtain the following inequality:

$$(u_h,u_h)_\Omega+\frac{\tau}{2}\sum_{K\in\Omega}\int\limits_{\Gamma(K)-\Gamma^*(K)} S_h u_h\cdot u_h ds+\frac{\tau}{2}\int\limits_{\Gamma(\Omega)} S_h u_h\cdot u_h ds\le 0.\tag{24}$$

It is obvious that

$$\sum_{K\in\Omega}\int\limits_{\Gamma(K)-\Gamma^*(K)} S_h u_h\cdot u_h ds\equiv 0.\tag{25}$$

Taking of equalities (25) from (24) we get the following:

$$(u_h,u_h)_\Omega\le 0.\tag{26}$$

From the last inequality it follows that $u_h \equiv 0$. According to Lemma 3.2 the following inequality is true:

$$\begin{aligned}&2(u_h^n,u_h^n)_K+\tau\int\limits_{\Gamma(K)-\Gamma^*(K)} S_h u_h^n\cdot u_h^n ds+\tau\int\limits_{\Gamma^*(K)} S_h u_h^n\cdot u_h^n ds\\&\le 2(u_h^{n-1}+\tau\cdot F_h^n,u_h^n)_K,\end{aligned}\tag{27}$$

where $u_h^k = u_h(t_k, x, y)$, $u_h^{k-1} = u_h(t_{k-1}, x, y)$ and $F_h^k = F_h(t_k, x, y)$. From (27) it follows that

$$\begin{aligned}2|u_h^n|_K^2 + \tau \int\limits_{\Gamma(K)-\Gamma^*(K)} S_h u_h^n \cdot u_h^n ds + \tau \int\limits_{\Gamma^*(K)} S_h u_h^n \cdot u_h^n ds \\ \leq 2(u_h^{n-1} + \tau \cdot F_h^n, u_h^n)_K \\ \leq \left\|u_h^{n-1} + \tau \cdot F_h^n\right\|_K^2 + \|u_h^n\|_K^2 . \end{aligned} \tag{28}$$

From the inequality (28) we obtain

$$\begin{aligned}|u_h^n|_K^2 + \tau \int\limits_{\Gamma(K)-\Gamma^*(K)} S_h u_h^n \cdot u_h^n ds + \tau \int\limits_{\Gamma^*(K)} S_h u_h^n \cdot u_h^n ds \\ \leq \left\|u_h^{n-1} + \tau \cdot F_h^n\right\|_K^2 \\ \leq \left(\left\|u_h^{n-1}\right\|_K + \tau\|F_h^n\|_K\right)^2 \\ \leq (\tau+1)\left\|u_h^{n-1}\right\|_K^2 + (\tau^2+\tau)\|F_h^n\|_K^2 . \end{aligned} \tag{29}$$

Taking into account the identity (25) and the condition (8), from inequality (29) it follows that

$$|u_h^k|_\Omega^2 \leq (\tau+1)\left\|u_h^{k-1}\right\|_\Omega^2 + (\tau^2+\tau)\left\|F_h^k\right\|_\Omega^2 . \tag{30}$$

From the last inequality we have:

$$\begin{aligned}|u_h(t,x,y)|_\Omega^2 \leq e^T \|u_h(0,x,y)\|_\Omega^2 \\ + (T+1)(e^T-1)F, \end{aligned} \tag{31}$$

where $F = \max\limits_{t\in[0,T]} \|F_h(t,x,y))\|_\Omega^2$. Theorem is proven.

4. Numerical Computations

For the numerical solution of the mixed problem (1)-(5) the complex programmes Delphi-7 is applied. For the solution of the obtained system of algebraic equations "The method of minimal discrepancy" is used. **Example 1:** As model problem in the domain

$$\Omega = \{(x,y) : 0 < x < 2,\ 0 < y < 2\}$$

we consider the following system

$$\begin{cases} \frac{\partial u_1}{\partial t} + y\frac{\partial u_1}{\partial x} + x\frac{\partial u_1}{\partial y} - u_2 = x^2 + y^2 \\ \qquad\qquad\qquad\qquad +x + y + 2t - t^2 - 4 \\ \frac{\partial u_2}{\partial t} - y\frac{\partial u_2}{\partial x} - x\frac{\partial u_2}{\partial y} + u_1 = t^2 + xy + x + y + 2t \end{cases}$$

with the boundary conditions

for $x = 0$: $u_1 = t^2$;

for $x = 2$: $u_2 = 2 - y + t^2$;

for $y = 0$: $u_2 = 2 - x + t^2$
and with the initial data for $t = 0$:

$$\begin{cases} u_1 = xy, \\ u_2 = 4 - x - y. \end{cases}$$

The exact solution of this mixed problem is as follows:

$$u_1 = xy + t^2; \; u_2 = 4 - x - y + t^2.$$

It is not difficult to check that above mentioned mixed problem satisfies conditions of Theorem 3.1. The values of the error is computed

$$\|u - v\|, \quad \|u\|_\Omega = \sqrt{\int_\Omega u \cdot u}$$

Table 1 given below show the error values of the numerical solution for the values of the parameters $N_x = 10, 20$; $N_y = 10, 20$; $t = 10$. Here v is the numerical solution of the mixed problem by the finite elements method

Table 1: Error values of Example 1

N_t	$N_x = 10, \; N_y = 10$	$N_x = 20, \; N_y = 20$
10	5.2580955	2.9865759
20	2.6233082	1.4951976
40	1.3014823	0.7477249
80	0.6395239	0.3731776
160	0.3107552	0.1865482

Table 1 shows that the error in the numerical solution tends to zero when the difference grid steps tend to zero. Since the error of approximation is of first order $O(\tau, h_x, h_y)$, this result is acceptable. Pay attention to the range of integration (at the $N_x = N_y = 20$, a step value is $h_x = h_y = 0.1$ and hence $O(\tau, h_x, h_y) = O(0.1)$).

Example 2: In the domain

$$\Omega = \{(x, y) : 0 < x < 2, \quad 0 < y < 2\}$$

we consider the following system

$$\begin{cases} \frac{\partial u_1}{\partial t} + y\frac{\partial u_1}{\partial x} + t\frac{\partial u_1}{\partial y} + yu_1 - xu_2 = xt(y-t) \\ \qquad\qquad +t^2 - x^3 + (x+y+y^2)(1+t), \\ \frac{\partial u_2}{\partial t} - t\frac{\partial u_2}{\partial x} + x\frac{\partial u_2}{\partial y} + xu_1 - yu_2 = 2(t+xy) \\ \qquad\qquad +xt(x+y-2) - y^3 + yt^2 \end{cases}$$

with the boundary conditions

for $x = 0$: $u_1 = yt$;

for $x = 2$: $u_2 = 4 + y^2 + t^2$;

for $y = 0$: $u_1 = xt$; $u_2 = x^2 + t^2$
and with the initial data for $t = 0$:

$$\begin{cases} u_1 = xy, \\ u_2 = x^2 + y^2. \end{cases}$$

The exact solution of this mixed problem is as follows:

$$u(x, y, t) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} xy + xt + yt \\ x^2 + y^2 + t^2 \end{pmatrix}$$

It is not difficult to check that above mentioned mixed problem satisfies conditions of Theorem 3.1. The error values of the numerical solution for the values of the parameters

$$n_t = 20, \quad n_x = 20, \quad n_y = 20$$

in $\|u\| = \sqrt{J(t)}$ is equal

$$\|u - v\| = 0.1201078.$$

Here v is the numerical solution of the mixed problem by the finite elements method.

5. Conclusion

In this paper, finite element method is used to find approximate solution of symmetric hyperbolic systems with variable coefficients. Stability of the finite element method is proved. Numerical examples indicates that the proposed method is highly efficient for the tested problem.

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